

THE MODULI SPACE OF TYPE \mathcal{A} SURFACES WITH TORSION AND NON-SINGULAR SYMMETRIC RICCI TENSOR

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ABSTRACT. We examine the moduli spaces of Type \mathcal{A} connections on oriented and unoriented surfaces both with and without torsion in relation to the signature of the associated symmetric Ricci tensor ρ_s . If the signature of ρ_s is $(1, 1)$ or $(0, 2)$, the spaces are smooth. If the signature is $(2, 0)$, there is an orbifold singularity.

1. INTRODUCTION

Let ∇ be a connection on the tangent bundle of a smooth surface M . Introduce a system of local coordinates $x := (x^1, x^2)$ on M to expand $\nabla_{\partial_{x^i}} \partial_{x^j} = \Gamma_{ij}^k \partial_{x^k}$ where we adopt the *Einstein convention* and sum over repeated indices. The components of the curvature operator R are given by:

$$R_{ijk}^l = \partial_{x_i} \Gamma_{jk}^l - \partial_{x_j} \Gamma_{ik}^l + \Gamma_{in}^l \Gamma_{jk}^n - \Gamma_{jn}^l \Gamma_{ik}^n.$$

We say that \mathcal{M} is *symmetric* if $\nabla R = 0$. Let $\rho(x, y) := \text{Tr}\{z \rightarrow R(z, x)y\}$ be the *Ricci tensor* and let $\nabla \rho$ be the covariant derivative of the Ricci tensor. Although the Ricci tensor is a symmetric 2-tensor in the Riemannian setting, it need not be symmetric in this more general setting and we define the *symmetric Ricci tensor* by setting $\rho_s(X, Y) := \frac{1}{2}\{\rho(X, Y) + \rho(Y, X)\}$. We say that $\mathcal{M} := (M, \nabla)$ is *locally homogeneous* if given any two points P and Q of M , there is the germ of a local diffeomorphism $\Phi_{P,Q}$ taking P to Q which commutes with ∇ .

We say that ∇ is *torsion free* if $\nabla_X Y - \nabla_Y X - [X, Y] = 0$; this symmetry is equivalent to the symmetry $\Gamma_{ij}^k = \Gamma_{ji}^k$. Torsion free connections on surfaces have been used to construct new examples of pseudo-Riemannian metrics exhibiting properties without Riemannian counterpart [8, 9, 10, 20].

The theory of connections with torsion plays an important role in string theory [1, 13, 15, 17], they are important in almost contact geometry [14, 18, 25, 28], they play a role in non-integrable geometries [1, 2, 3, 7], they are important in spin geometries [19], they are useful in considering almost hypercomplex geometries [24], they appear in the study of compact solvmanifolds [12], and they have been used to study the non-commutative residue for manifolds with boundary [29]. The following result was first proved in the torsion free setting by Opozda [26] and subsequently extended to surfaces with torsion by Arias-Marco and Kowalski [4], see also [4, 11, 16, 21, 22, 27] for related work.

Theorem 1.1. *Let $\mathcal{M} = (M, \nabla)$ be a locally homogeneous surface where ∇ can have torsion. Then at least one of the following three possibilities, which are not exclusive, hold which describe the local geometry:*

- (A) *There exists a coordinate atlas so the Christoffel symbols Γ_{ij}^k are constant.*
- (B) *There exists a coordinate atlas so the Christoffel symbols have the form $\Gamma_{ij}^k = (x^1)^{-1} C_{ij}^k$ for C_{ij}^k constant and $x^1 > 0$.*

2010 *Mathematics Subject Classification.* 53C21.

Key words and phrases. Ricci tensor, moduli space, locally homogeneous affine surface, connection with torsion, orbifold singularity.

(C) ∇ is the Levi-Civita connection of a metric of constant Gauss curvature.

Such a surface which is not flat is said to be of Type- \mathcal{A} , Type- \mathcal{B} , or Type- \mathcal{C} depending upon which of the possibilities hold in this result. These classes are not disjoint. While there are no surfaces which are both Type- \mathcal{A} and Type- \mathcal{C} , there are surfaces which are both Type- \mathcal{A} and Type- \mathcal{B} and there are surfaces which are both Type- \mathcal{B} and Type- \mathcal{C} . We refer to the discussion in [5] for further details.

We shall work in the Type \mathcal{A} setting for the remainder of this paper and shall let $\mathcal{M} := (M, \nabla)$ be a locally homogeneous Type \mathcal{A} surface with torsion. In this setting, the Christoffel symbols

$$\Gamma = \{\Gamma_{11}^1, \Gamma_{11}^2, \Gamma_{12}^1, \Gamma_{21}^1, \Gamma_{12}^2, \Gamma_{21}^2, \Gamma_{22}^1, \Gamma_{22}^2\}$$

belong to \mathbb{R}^8 . For $p + q = 2$, let $\mathcal{W}(p, q) \subset \mathbb{R}^8$ be the open subset of Christoffel symbols defining a Type \mathcal{A} structure such that the associated symmetric Ricci tensor $\rho_{s, \Gamma}$ is non-degenerate and of signature (p, q) ; in contrast to the torsion free setting, the Ricci tensor can be non-symmetric and thus it is necessary to deal with the symmetrization which was defined previously. Let $\mathfrak{W}^+(p, q)$ (resp. $\mathfrak{W}(p, q)$) be the corresponding moduli space of oriented (resp. unoriented) Type \mathcal{A} structures.

The general linear group $\text{GL}(2, \mathbb{R})$ acts on $\mathcal{W}(p, q)$ and on $\mathcal{Z}(p, q)$ in the obvious fashion by changing the basis for \mathbb{R}^2 ; we shall denote this action by $g\Gamma$. Let $\text{GL}^+(2, \mathbb{R}) \subset \text{GL}(2, \mathbb{R})$ be the connected component of the identity; these are the matrices with positive determinant. The action of the general linear group completely determines the isomorphism type in this setting.

Theorem 1.2. $\mathfrak{Z}^+(p, q) = \mathcal{Z}(p, q) / \text{GL}^+(2, \mathbb{R})$, $\mathfrak{W}^+(p, q) = \mathcal{W}(p, q) / \text{GL}^+(2, \mathbb{R})$,
 $\mathfrak{Z}(p, q) = \mathcal{Z}(p, q) / \text{GL}(2, \mathbb{R})$, $\mathfrak{W}(p, q) = \mathcal{W}(p, q) / \text{GL}(2, \mathbb{R})$.

Proof. Let $\mathcal{M} = (M, \nabla)$ be a Type \mathcal{A} surface with non-degenerate symmetric Ricci tensor. Choose local coordinate charts for M where the Christoffel symbols Γ_{ij}^k of ∇ are locally constant. Then

$$R_{ijk}^l = \Gamma_{in}^l \Gamma_{jk}^n - \Gamma_{jn}^l \Gamma_{ik}^n, \quad \rho_{jk} = \Gamma_{in}^i \Gamma_{jk}^n - \Gamma_{jn}^i \Gamma_{ik}^n, \\ \rho_{s, jk} = \frac{1}{2} \{ \Gamma_{in}^i \Gamma_{jk}^n - \Gamma_{jn}^i \Gamma_{ik}^n + \Gamma_{in}^i \Gamma_{kj}^n - \Gamma_{kn}^i \Gamma_{ij}^n \}.$$

The symmetric Ricci tensor gives an invariantly defined flat pseudo-Riemannian metric. Therefore, the transition functions between such charts are affine. The desired result follows since translations do not affect Γ . \square

There is an exceptional structure. Define $\Gamma_0 \in \mathcal{Z}(2, 0)$ and $C_0 \subset \mathcal{Z}(2, 0)$ by setting:

$$\Gamma_{0;11}^1 = -1, \Gamma_{0;12}^2 = \Gamma_{0;21}^2 = \Gamma_{0;22}^1 = 1, \Gamma_{0;ij}^k = 0 \text{ otherwise,} \\ C_0 := \Gamma_0 \cdot \text{GL}(2, \mathbb{R}) = \Gamma_0 \cdot \text{GL}^+(2, \mathbb{R}) \subset \mathcal{Z}(2, 0). \quad (1.a)$$

We will show presently in Lemma 2.2 that C_0 is a closed set and, furthermore, that any non-trivial fixed point of the action of $\text{GL}^+(2, \mathbb{R})$ on $\mathcal{Z}(p, q)$ belongs to this orbit; thus in particular, $\text{GL}^+(2, \mathbb{R})$ acts without fixed points on $\mathcal{Z}(1, 1)$ and on $\mathcal{Z}(0, 2)$. To exclude this exceptional orbit, we define:

$$\tilde{\mathcal{Z}}(p, q) := \begin{cases} \mathcal{Z}(p, q) & \text{if } (p, q) \neq (2, 0) \\ \mathcal{Z}(p, q) - C_0 & \text{if } (p, q) = (2, 0) \end{cases}, \\ \tilde{\mathcal{W}}(p, q) := \begin{cases} \mathcal{W}(p, q) & \text{if } (p, q) \neq (2, 0) \\ \mathcal{W}(p, q) - C_0 & \text{if } (p, q) = (2, 0) \end{cases}, \\ \tilde{\mathfrak{Z}}^+(p, q) := \tilde{\mathcal{Z}}(p, q) / \text{GL}^+(2, \mathbb{R}), \quad \tilde{\mathfrak{Z}}(p, q) := \tilde{\mathcal{Z}}(p, q) / \text{GL}(2, \mathbb{R}), \\ \tilde{\mathfrak{W}}^+(p, q) := \tilde{\mathcal{W}}(p, q) / \text{GL}^+(2, \mathbb{R}), \quad \tilde{\mathfrak{W}}(p, q) := \tilde{\mathcal{W}}(p, q) / \text{GL}(2, \mathbb{R}).$$

Theorem 1.3.

- (1) $\tilde{\mathfrak{Z}}^+(p, q)$ (resp. $\tilde{\mathfrak{W}}^+(p, q)$) is a smooth manifold of dimension 2 (resp. 4) and $\tilde{\mathcal{Z}}(p, q) \rightarrow \tilde{\mathfrak{Z}}^+(p, q)$ (resp. $\tilde{\mathcal{W}}(p, q) \rightarrow \tilde{\mathfrak{W}}^+(p, q)$) is a $\mathrm{GL}^+(2, \mathbb{R})$ principal bundle.
- (2) $\tilde{\mathfrak{Z}}(p, q)$ (resp. $\tilde{\mathfrak{W}}(p, q)$) is a smooth manifold with boundary (resp. without boundary) of dimension 2 (resp. 4). Furthermore, $\tilde{\mathfrak{Z}}^+(p, q) \rightarrow \tilde{\mathfrak{Z}}(p, q)$ (resp. $\tilde{\mathfrak{W}}^+(p, q) \rightarrow \tilde{\mathfrak{W}}(p, q)$) is a ramified double cover.

Section 2 is devoted to the proof of Assertion 2 and Section 3 is devoted to the proof of Assertion 3. The analysis is quite different in the unoriented context as the full general linear group $\mathrm{GL}(2, \mathbb{R})$ has a 1-dimensional fixed point set acting on $\mathcal{Z}(p, q)$ and a 2-dimensional fixed point set acting on $\mathcal{W}(p, q)$ for $(p, q) \in \{(2, 0), (1, 1), (0, 2)\}$. We will discuss the ramification sets in Section 3.3 and in Section 3.5 once the necessary notation has been developed. We must now consider the singular orbit $[\Gamma_0]$ when $(p, q) = (2, 0)$.

Definition 1.4. Let $\mathbb{Z}_3 := \{1, \lambda, \lambda^2\}$ for $\lambda := e^{2\pi\sqrt{-1}/3}$ be the cyclic group of order 3 consisting of the third roots of unity in \mathbb{C} . This group acts on \mathbb{C} and on \mathbb{C}^2 by complex multiplication. Since $\bar{\lambda} = \lambda^2$, complex conjugation defines a non-trivial automorphism of the group \mathbb{Z}_3 . Thus complex multiplication by λ and complex conjugation generate a non-Abelian group of order 6 which is isomorphic to the symmetric group s_3 which consists of the permutations of 3 elements. This group acts naturally on \mathbb{C} and on \mathbb{C}^2 ; \mathbb{Z}_3 is a normal subgroup of s_3 .

We will establish the following result in Section 4.

Theorem 1.5.

- (1) $\mathfrak{Z}^+(2, 0)$ (resp. $\mathfrak{W}^+(2, 0)$) is a smooth orbifold. An orbifold coordinate chart near $[\Gamma_0]$ can be obtained by taking \mathbb{C} (resp. \mathbb{C}^2) modulo the action of \mathbb{Z}_3 .
- (2) $\mathfrak{Z}(2, 0)$ (resp. $\mathfrak{W}(2, 0)$) is a smooth orbifold. An orbifold coordinate chart near $[\Gamma_0]$ can be defined by taking \mathbb{C} (resp. \mathbb{C}^2) modulo the action of s_3 .

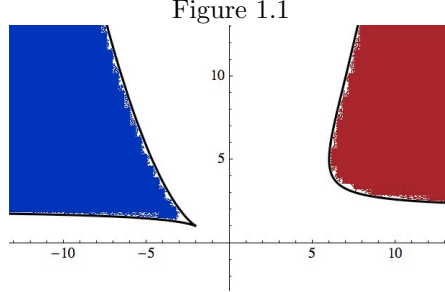
We note that although we shall work in the smooth category, these results continue to hold in the real analytic category. We now turn to the question of invariants and, for the sake of completeness, present some results from M. Brozos-Vázquez et al. [6]. We work in the torsion free setting. If $\Gamma \in \mathcal{Z}(p, q)$, let dvol be the oriented 2-form that gives the pseudo-Riemannian volume element defined by ρ , let $\rho_{ij}^3 := \Gamma_{ik}^l \Gamma_{jl}^k$, let $\psi_3 := \mathrm{Tr}_\rho\{\rho^3\} = \rho^{ij} \rho_{ij}^3$, let $\Psi_3 := \det(\rho^3)/\det(\rho)$, and let $\chi(\Gamma) := \rho(\Gamma_{ab}^b \Gamma_{ij}^k \rho_{kl}^3 \rho^{ij} dx^a \wedge dx^l, \mathrm{dvol})$.

Theorem 1.6. Let $\mathcal{M} = (M, \nabla)$ be a locally homogeneous torsion free surface of Type \mathcal{A} where $\mathrm{Rank}\{\rho\} = 2$.

- (1) ψ_3 , Ψ_3 , and χ are invariantly defined on \mathcal{M} and does not depend on the particular choice of Type \mathcal{A} coordinates.
- (2) $\Xi(p, q) := (\psi_3, \Psi_3, \chi)$ is a 1-1 map from each $\mathfrak{Z}^+(p, q)$ to a closed surface in \mathbb{R}^3 and provides a complete set of invariants in the oriented context.
- (3) $\Theta(p, q) := (\psi_3, \Psi_3)$ defines a 1-1 map from $\mathfrak{Z}(p, q)$ to a simply connected closed subset $\mathfrak{V}(p, q)$ of \mathbb{R}^2 and provides a complete set of invariants in the unoriented context.

We wish to make this description of the moduli spaces as specific as possible. Consider the curves $\sigma_\pm(t) := (\pm 4t^2 \pm \frac{1}{t^2} + 2, 4t^4 \pm 4t^2 + 2)$. The curve σ_+ is smooth; the curve σ_- has a cusp at $(-2, 1)$ when $t = \frac{1}{\sqrt{2}}$. This corresponds to the structure Γ_0 given above in Equation (1.a). The curves σ_\pm divide the plane into 3 open regions $\mathfrak{D}(2, 0)$, $\mathfrak{D}(1, 1)$, and $\mathfrak{D}(0, 2)$. The region $\mathfrak{D}(2, 0)$ lies in the second quadrant and is bounded on the right by σ_- ; the region $\mathfrak{D}(0, 2)$ lies in the first quadrant and is

bounded on the left by σ_+ ; the region $\mathfrak{D}(1,1)$ lies in between and is bounded on the left by σ_- and on the right by σ_+ . The regions $\mathfrak{D}(p,q) = \Theta(p,q)\{\mathfrak{Z}(p,q)\}$ discussed in Theorem 1.6 are the closure of the open sets $\mathfrak{D}(p,q)$. We picture below $\mathfrak{D}(2,0)$ in blue, $\mathfrak{D}(1,1)$ in white, and $\mathfrak{D}(0,2)$ in red:



Note that although $\Theta(p,q)$ is 1-1 on $\mathfrak{Z}(p,q)$, $\Theta(0,2)(\mathfrak{Z}(0,2))$ intersects $\Theta(1,1)(\mathfrak{Z}(1,1))$ along their common boundary σ_+ and $\Theta(2,0)(\mathfrak{Z}(2,0))$ intersects $\Theta(1,1)(\mathfrak{Z}(1,1))$ along their common boundary σ_- . This does not mean that $\mathfrak{Z}(2,0)$ or $\mathfrak{Z}(0,2)$ intersects $\mathfrak{Z}(1,1)$ nor does it mean that $\Theta(p,q)$ is not 1-1 on these sets separately. Although it appears from the picture that $\mathfrak{Z}(1,1)$ has a cusp singularity, Theorem 1.3 shows that this is not the case and the apparent cusp is an artifact of the parametrization in terms of the given invariants. On the other hand, $\mathfrak{Z}(2,0)$ really does have a singularity at Γ_0 by Theorem 1.5.

2. PROPER AND FIXED POINT FREE GROUP ACTIONS

2.1. Principal bundles. Let G be a Lie group which acts smoothly on a manifold N . One says the action of G on N is *proper* if given $P_n \in N$ and $g_n \in G$ with $P_n \rightarrow P$ and $g_n P_n \rightarrow \tilde{P}$, there is a subsequence so $g_{n_k} \rightarrow g \in G$. If $P \in N$, let $G_P := \{g \in G : gP = P\}$ be the isotropy subgroup. One says the action is *fixed point free* if $G_P = \{\text{id}\}$ for any $P \in N$.

Lemma 2.1. *Let G be a Lie group acting smoothly on a smooth manifold N .*

- (1) *If G_P is a discrete subgroup of G , then $\Psi_P : g \rightarrow g \cdot P$ is a smooth immersion of G into N .*
- (2) *If the action of G on N is proper and fixed point free, then N/G is a smooth manifold and $N \rightarrow N/G$ is a principal G bundle.*

Proof. Although this result is well-known, we sketch the proof briefly to introduce the notation we shall need subsequently. Let $\mathfrak{g} := T_{\text{id}}(G)$ be the Lie algebra of G . Let $\exp : \mathfrak{g} \rightarrow G$ be the exponential map. If $0 \neq \xi \in \mathfrak{g}$, let $\gamma_{\xi,P}(t) := \exp(t\xi) \cdot P$. We wish to show that $\dot{\gamma}_{\xi,P}(0) \neq 0$; we suppose to the contrary that $\dot{\gamma}_{\xi,P}(0) = 0$ and argue for a contradiction. Since $\exp((t+s)\xi) = \exp(t\xi)\exp(s\xi)$, one has that $\gamma_{\xi,P}(t+s) = \exp(t\xi) \cdot \gamma_{\xi,P}(s)$ and $\dot{\gamma}_{\xi,P}(t) = \exp(t\xi)_* \dot{\gamma}_{\xi,P}(0) = 0$. Since $\dot{\gamma}_{\xi,P}(t) = 0$ for all t , $\gamma_{\xi,P}(t)$ is the constant map. This contradicts the hypothesis that G_P is a discrete subgroup of G . Thus we conclude that $\dot{\gamma}_{\xi,P}(0) \neq 0$ so $d\Psi_P(0)$ is an injective map from \mathfrak{g} to $T_P N$. This shows that Ψ_P is an immersion near $g = \text{id}$. We use the group action to see that Ψ_P is an immersion near any point of G .

Suppose the action is fixed point free and proper. Let Σ be the germ of a submanifold which is transverse to the orbit $\mathcal{O}_P := G \cdot P$ at P ; Σ is called a *slice*. Let $\Phi(g,s) := g \cdot s$ for $g \in G$ and $s \in \Sigma$. Since Σ is transverse to \mathcal{O}_P at P , Assertion 1 implies that Ψ_* is an isomorphism from $T_{\text{id}}(G) \times T_P(\Sigma)$ to $T_P N$. We use the transitive group action on \mathcal{O}_P to see that Φ is an immersion from $G \times \Sigma \rightarrow N$. We wish to show that Φ is an embedding if we restrict to a sufficiently small neighborhood of P in Σ . Suppose, to the contrary, that no such neighborhood

exists. This implies that there exists $(g_n, P_n, \tilde{g}_n, \tilde{P}_n)$ so that

$$\begin{aligned} P_n &\in \Sigma, & \tilde{P}_n &\in \Sigma, & g_n &\in G, & \tilde{g}_n &\in G, \\ P_n &\rightarrow P, & \tilde{P}_n &\rightarrow P, & g_n P_n &= \tilde{g}_n \tilde{P}_n, & (g_n, P_n) &\neq (\tilde{g}_n, \tilde{P}_n). \end{aligned}$$

By replacing g_n by $\tilde{g}_n^{-1}g_n$, we may assume $\tilde{g}_n = 1$. Since the action is proper, we can choose a subsequence of the g_n which converges to g . Since $P_n \rightarrow P$ and $\tilde{P}_n \rightarrow P$, we have $gP = P$. Since the action is fixed point free, $g = \text{id}$ and thus $g_n \rightarrow \text{id}$. This contradicts the fact that $G \cdot \Sigma \rightarrow N$ is an immersion. Thus if we restrict Σ suitably, $G \times \Sigma$ may be identified with a suitable closed subset of N which is a neighborhood of $G \cdot P$. This gives the requisite principal bundle charts; the projection of Σ to N/G is 1-1 and gives the appropriate charts on the quotient N/G ; the transition maps between these charts are smooth. \square

2.2. The action of $\text{GL}^+(2, \mathbb{R})$ on $\mathcal{Z}(p, q)$. If $\Gamma \in \mathcal{Z}(p, q)$, set:

$$\text{SO}^+(\varrho_{s,\Gamma}) := \{T \in \text{GL}^+(2, \mathbb{R}) : T^* \varrho_{s,\Gamma} = \varrho_{s,\Gamma}\},$$

$$G_\Gamma^+ := \{T \in \text{GL}^+(2, \mathbb{R}) : TT = \Gamma\} \subset \text{SO}^+(\varrho_{s,\Gamma}).$$

Lemma 2.2. *Let $p + q = 2$. Let Γ_0 be the exceptional structure of Equation (1.a).*

- (1) *The action of $\text{GL}(2, \mathbb{R})$ on $\mathcal{W}(p, q)$ and on $\mathcal{Z}(p, q)$ is proper*
- (2) *$C_0 := \text{GL}^+(2, \mathbb{R}) \cdot \Gamma_0$ is a closed subset of $\mathcal{Z}(2, 0)$ and of $\mathcal{W}(2, 0)$.*
- (3) *If $\Gamma \in \mathcal{Z}(p, q)$ satisfies $G_\Gamma^+ \neq \{\text{id}\}$, then $(p, q) = (2, 0)$ and $\Gamma \in C_0$.*

Proof. We wish to prove the action is proper. Since $\text{GL}^+(2, \mathbb{R})$ is a subgroup of finite index in $\text{GL}(2, \mathbb{R})$, it suffices to prove the action of $\text{GL}^+(2, \mathbb{R})$ on $\mathcal{W}(p, q)$ is proper; it will then follow that the action of $\text{GL}(2, \mathbb{R})$ is proper. Restricting to $\mathcal{Z}(p, q)$ then yields a proper action as well.

Suppose that there exists $g_n \in \text{GL}^+(2, \mathbb{R})$ and that there exist Γ_n, Γ , and $\tilde{\Gamma}$ in $\mathcal{W}(p, q)$ so that $\Gamma_n \rightarrow \Gamma$ and $g_n \Gamma_n \rightarrow \tilde{\Gamma}$. To prove Assertion 1, we must extract a subsequence g_{n_k} which is convergent. Let $\varrho := \rho_{s,\tilde{\Gamma}}$; by hypothesis, ϱ is a non-degenerate symmetric bilinear form of signature (p, q) . Let $\tilde{\Gamma}_n := g_n \Gamma_n$. Since $\tilde{\Gamma}_n \rightarrow \tilde{\Gamma}$, we have $\rho_{s,\tilde{\Gamma}_n} \rightarrow \varrho$. For n large, we can apply the Gram-Schmidt process to find $h_n \in \text{GL}^+(2, \mathbb{R})$ with $h_n \rightarrow \text{id}$ and $h_n \rho_{s,\tilde{\Gamma}_n} = \varrho$; it is necessary to take n large to ensure $\rho_{s,\tilde{\Gamma}_n}$ is close to ϱ and thus we are not dividing by zero when applying the Gram-Schmidt process as ϱ could be indefinite. We then have $\rho_{h_n g_n \Gamma_n} = \varrho$ and $h_n g_n \Gamma_n \rightarrow \tilde{\Gamma}$. Since extracting a convergent sequence from $h_n g_n$ is equivalent to extracting a convergent sequence from g_n , we may assume without loss of generality that $\rho_{s,g_n \Gamma_n} = \varrho$ for all n .

We have also that $\tilde{\Gamma}_n \rightarrow \tilde{\Gamma}$ and $g_n^{-1} \tilde{\Gamma}_n \rightarrow \Gamma$. Choose h so $h \rho_\Gamma = \varrho$. Extracting a convergent sequence from amongst the g_n is equivalent to extracting a convergent sequence from amongst the $h g_n^{-1}$. Thus we may assume $\rho_{s,\Gamma} = \varrho$ as well without altering the normalizations $\rho_{\tilde{\Gamma}_n} = \rho_{\tilde{\Gamma}} = \varrho$. We clear the previous notation and apply the Gram-Schmidt process to find $h_n \in \text{GL}^+(2, \mathbb{R})$ so $h_n \rightarrow \text{id}$ and $h_n \rho_{g_n^{-1} \tilde{\Gamma}_n} = \varrho$. Replacing the g_n^{-1} by $h_n g_n^{-1}$, we may assume

$$\rho_\Gamma = \varrho \quad \rho_{\tilde{\Gamma}} = \varrho, \quad \rho_{\Gamma_n} = \varrho, \quad \rho_{g_n \Gamma_n} = \varrho.$$

This implies that the $g_n \in \text{SO}^+(\varrho)$. If $(p, q) = (2, 0)$ or $(p, q) = (0, 2)$, then $\text{SO}^+(\varrho)$ is a compact Lie group and we can extract a convergent subsequence. We therefore assume $(p, q) = (1, 1)$.

Choose the basis for \mathbb{R}^2 so $\varrho = dx^1 \otimes dx^2 + dx^2 \otimes dx^1$. Since $g_n \in \text{SO}^+(\varrho)$, we may conclude $g_n(x^1, x^1) = (a_n x^1, a_n^{-1} x^2)$. If $|a_n|$ and $|a_n^{-1}|$ remain uniformly bounded, we can extract a convergent subsequence. Thus by interchanging the roles of x^1

and x^2 if necessary, we may assume that $|a_n| \rightarrow \infty$. We argue for a contradiction. We express

$$(g_n \Gamma)_{n,ij}{}^k = a_n^{\epsilon_{ijk}} \Gamma_{n,ij}{}^k \text{ for} \quad (2.a)$$

$$\epsilon_{ijk} := \delta_{1i} - \delta_{2i} + \delta_{1j} - \delta_{2j} - \delta_{1k} + \delta_{2k} \in \{\pm 1, \pm 3\}.$$

Since $(g_n \Gamma)_{ij}{}^k$ converges to $\tilde{\Gamma}_{ij}{}^k$ and $\Gamma_{n,ij}{}^k$ converge to $\Gamma_{ij}{}^k$, we have $\Gamma_{ij}{}^k = 0$ for $\epsilon_{ijk} > 0$. Consequently, $\Gamma_{11}{}^1 = 0$, $\Gamma_{11}{}^2 = 0$, $\Gamma_{12}{}^2 = 0$ and $\Gamma_{21}{}^2 = 0$. We may then compute that $\rho_{11} = \rho_{12} = \rho_{21} = 0$ which is impossible. This contradiction completes the proof of Assertion 1. Since the action by $\text{GL}^+(2, \mathbb{R})$ is proper, any orbit is closed. This establishes Assertion 2.

To prove Assertion 3, we examine the isotropy group. Assume that there exists $\text{id} \neq g \in G_\Gamma^+$. Since $g\Gamma = \Gamma$, $g\rho_{s,\Gamma} = \rho_{s,\Gamma}$ so $g \in \text{SO}^+(\rho_{s,\Gamma})$. If $\rho_{s,\Gamma}$ has indefinite signature, then we can choose the coordinates so $gx^1 = ax^1$ and $gx^2 = a^{-1}x^1$ for $a \neq 1$. Adopt the notation of Equation (2.a). We have $(g\Gamma)_{ij}{}^k = a^{\epsilon_{ijk}} \Gamma_{ij}{}^k$. Since ϵ_{ijk} is odd and $a \neq 1$, setting $g\Gamma = \Gamma$ implies all the $\Gamma_{ij}{}^k$ vanish which is false. Thus the action is fixed point free in signature (1, 1).

Suppose the signature is definite. Introduce a complex basis

$$\begin{aligned} f_1 &:= e_1 + \sqrt{-1}e_2, & f_2 &:= e_1 - \sqrt{-1}e_2, & gf_1 &= \alpha f_1, & gf_2 &= \bar{\alpha} f_2, \\ f^1 &:= \frac{1}{2}(e^1 - \sqrt{-1}e^2), & f^2 &:= \frac{1}{2}(e^1 + \sqrt{-1}e^2) & gf^1 &= \bar{\alpha} f^1, & gf^2 &= \alpha f^2 \end{aligned} \quad (2.b)$$

for $\alpha = e^{\sqrt{-1}\theta}$ appropriately chosen to describe the rotation involved. Since $\text{id} \neq g$, $\alpha \neq 1$. Let $\tilde{\Gamma}_{ij}{}^k$ reflect the Christoffel symbols relative to this complex basis. Adopt the notation of Equation (2.a). We then have $(g\tilde{\Gamma})_{ij}{}^k = \alpha^{\epsilon_{ijk}} \tilde{\Gamma}_{ij}{}^k$. This implies $\tilde{\Gamma}_{ij}{}^k = 0$ if $\epsilon_{ijk} = \pm 1$ and furthermore, that $\alpha^3 = 0$. Thus there is a complex number β so

$$\begin{aligned} \tilde{\Gamma}_{11}{}^1 &= 0, & \tilde{\Gamma}_{11}{}^2 &= \beta, & \tilde{\Gamma}_{12}{}^1 &= 0, & \tilde{\Gamma}_{12}{}^2 &= 0, \\ \tilde{\Gamma}_{21}{}^1 &= 0, & \tilde{\Gamma}_{21}{}^2 &= 0, & \tilde{\Gamma}_{22}{}^1 &= \bar{\beta}, & \tilde{\Gamma}_{22}{}^2 &= 0. \end{aligned}$$

Since $\tilde{\Gamma}_{12}{}^1 = \tilde{\Gamma}_{21}{}^1$ and $\tilde{\Gamma}_{12}{}^2 = \tilde{\Gamma}_{21}{}^2$, $\tilde{\Gamma}$ and hence Γ is torsion free. By performing a coordinate rotation, we can assume that β is real. We obtain

$$\begin{aligned} 0 &= \tilde{\Gamma}_{11}{}^1 = \frac{1}{2}\{\Gamma_{11}{}^1 + 2\Gamma_{12}{}^2 - \Gamma_{22}{}^1 + \sqrt{-1}(-\Gamma_{11}{}^2 + 2\Gamma_{12}{}^1 + \Gamma_{22}{}^2)\}, \\ 0 &= \tilde{\Gamma}_{12}{}^1 = \frac{1}{2}\{\Gamma_{11}{}^1 + \Gamma_{22}{}^1 + \sqrt{-1}(-\Gamma_{11}{}^2 - \Gamma_{22}{}^2)\}, \\ \beta &= \tilde{\Gamma}_{11}{}^2 = \frac{1}{2}\{\Gamma_{11}{}^1 - 2\Gamma_{12}{}^2 - \Gamma_{22}{}^1 + \sqrt{-1}(+\Gamma_{11}{}^2 + 2\Gamma_{12}{}^1 - \Gamma_{22}{}^2)\}. \end{aligned}$$

We solve these equations to obtain the relations

$$\begin{aligned} \Gamma_{11}{}^1 &= \frac{1}{2}\beta, & \Gamma_{22}{}^1 &= -\frac{1}{2}\beta, & \Gamma_{12}{}^2 &= \Gamma_{21}{}^2 = -\frac{1}{2}\beta, \\ \Gamma_{11}{}^2 &= 0, & \Gamma_{22}{}^2 &= 0, & \Gamma_{12}{}^1 &= \Gamma_{21}{}^1 = 0. \end{aligned}$$

By rescaling the coordinate system, we can ensure $\frac{1}{2}\beta = -\frac{1}{\sqrt{2}}$ and obtain the structure of Equation (1.a):

$$\begin{aligned} \Gamma_{11}{}^1 &= -\frac{1}{\sqrt{2}}, & \Gamma_{22}{}^1 &= \frac{1}{\sqrt{2}}, & \Gamma_{12}{}^2 &= \Gamma_{21}{}^2 = \frac{1}{\sqrt{2}}, \\ \Gamma_{11}{}^2 &= 0, & \Gamma_{22}{}^2 &= 0, & \Gamma_{12}{}^1 &= \Gamma_{21}{}^1 = 0. \end{aligned}$$

One may then compute that $\rho_s = \text{diag}(-1, -1)$ which has signature (2, 0). This completes the proof of Assertion 3. \square

2.3. The proof of Theorem 1.3 Assertion 1. This is a direct consequence of Lemma 2.1 and of Lemma 2.2. \square

3. THE PROJECTIONS $\tilde{\mathfrak{Z}}^+(p, q) \rightarrow \tilde{\mathfrak{Z}}(p, q)$ AND $\tilde{\mathfrak{W}}^+(p, q) \rightarrow \tilde{\mathfrak{W}}(p, q)$

3.1. The nature of the ramification set. Fix an element $T \in \mathrm{GL}(2, \mathbb{R})$ of order 2 with $\det(T) = -1$. If $g \in \mathrm{GL}^+(2, \mathbb{R})$, twist the standard action by defining $g * \Gamma := TgT^{-1}\Gamma$. The map $\Gamma \rightarrow T\Gamma$ then intertwines these actions and consequently T descends to define a map $[T]$ of order 2 on the moduli spaces which defines an action of \mathbb{Z}_2 so that

$$\mathfrak{Z}(p, q) = \tilde{\mathfrak{Z}}^+(p, q)/\mathbb{Z}_2 \text{ and } \mathfrak{W}(p, q) = \tilde{\mathfrak{W}}^+(p, q)/\mathbb{Z}_2.$$

Denote the fixed point sets by:

$$\begin{aligned} \widetilde{\mathfrak{Z}}^+(p, q) &:= \{[\Gamma] \in \tilde{\mathfrak{Z}}^+(p, q) : [T\Gamma] = [\Gamma]\}, \\ \widetilde{\mathfrak{W}}^+(p, q) &:= \{[\Gamma] \in \tilde{\mathfrak{W}}^+(p, q) : [T\Gamma] = [\Gamma]\}. \end{aligned}$$

On the complement of the fixed point set, the projection from the oriented moduli space to the unoriented moduli space is a double cover and the \mathbb{Z}_2 quotient inherits a natural smooth structure. We examine the fixed point sets as follows:

Lemma 3.1.

- (1) $\widetilde{\mathfrak{Z}}^+(p, q)$ is a smooth 1-dimensional submanifold of $\tilde{\mathfrak{Z}}^+(p, q)$.
- (2) $\widetilde{\mathfrak{W}}^+(p, q)$ is a smooth 2-dimensional submanifold of $\tilde{\mathfrak{W}}^+(p, q)$.

Proof. We have excluded the exceptional orbit of Γ_0 and restrict to the oriented moduli spaces; $[T]$ is then smooth. By averaging over the action of \mathbb{Z}_2 , we can put smooth Riemannian metrics on the oriented moduli spaces. The fixed point sets are then totally geodesic submanifolds of $\tilde{\mathfrak{Z}}^+(p, q)$ and $\tilde{\mathfrak{W}}^+(p, q)$, respectively. In particular they are smooth.

Let $Te_1 = -e_1$ and $Te_2 = e_2$. Then $T\Gamma = \Gamma$ implies $\Gamma = \Gamma(a, b, c, d)$ where:

$$\begin{aligned} \Gamma_{11}^1 &= 0, & \Gamma_{11}^2 &= a, & \Gamma_{12}^1 &= b, & \Gamma_{12}^2 &= 0, \\ \Gamma_{21}^1 &= c, & \Gamma_{21}^2 &= 0, & \Gamma_{22}^1 &= 0, & \Gamma_{22}^2 &= d, \end{aligned}$$

We exclude the exceptional orbit $C_0 := \Gamma_0 \cdot \mathrm{GL}(2, \mathbb{R})$ and define:

$$\begin{aligned} \widetilde{\mathcal{FZ}}(p, q) &:= \{\Gamma(a, c, c, d) \in \tilde{\mathcal{Z}}(p, q)\} \cap C_0^c, \\ \widetilde{\mathcal{FW}}(p, q) &:= \{\Gamma(a, b, c, d) \in \tilde{\mathcal{W}}(p, q)\} \cap C_0^c; \end{aligned}$$

$\widetilde{\mathcal{FZ}}(p, q)$ is an open subset of \mathbb{R}^3 and $\widetilde{\mathcal{FW}}(p, q)$ is an open subset of \mathbb{R}^4 since

$$\rho_\Gamma = \begin{pmatrix} a(d-c) & 0 \\ 0 & b(c-d) \end{pmatrix}. \quad (3.a)$$

If $T\Gamma_1 = \Gamma_1$, $T\Gamma_2 = \Gamma_2$, and $g\Gamma_1 = \Gamma_2$ for $g \in \mathrm{GL}^+(2, \mathbb{R})$ for $\Gamma_i \in \widetilde{\mathcal{FW}}(p, q)$, then

$$TgT\Gamma_1 = Tg\Gamma_1 = T\Gamma_2 = \Gamma_2 \text{ so } g^{-1}TgT\Gamma_1 = \Gamma_1.$$

Since we have excluded the exceptional orbit from consideration, $\mathrm{GL}^+(2, \mathbb{R})$ acts without fixed points and $Tg = gT$. Thus the structure group in question is given by $G_0 := \{g \in \mathrm{GL}^+(2, \mathbb{R}) : Tg = gT\}$. We compute:

$$\begin{aligned} Tg &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} -a_{11} & -a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\ &= gT = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -a_{11} & a_{12} \\ -a_{21} & a_{22} \end{pmatrix}. \end{aligned}$$

This implies $a_{12} = a_{21} = 0$ so

$$G_0 = \left\{ g = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} : \det(g) > 0 \right\}$$

is two dimensional. By Lemma 2.2, the action of G_0 on $\widetilde{\mathcal{FZ}}(p, q)$ and on $\widetilde{\mathcal{FW}}(p, q)$ is proper and without fixed points. Thus the projections to the oriented moduli spaces are principal G_0 bundles. It then follows that

$$\dim\{\widetilde{\mathfrak{Z}}^+(p, q)\} = 3 - 2 = 1 \text{ and } \dim\{\widetilde{\mathfrak{W}}^+(p, q)\} = 4 - 2 = 2. \quad \square$$

3.2. The proof of Theorem 1.3 Assertion 2. Let F be a component of the fixed point set and let ν be the normal bundle of F . The \mathbb{Z}_2 action acts as multiplication by -1 on ν . If F has codimension 1, this replaces the open fiber intervals $(-\varepsilon, \varepsilon)$ by half open intervals $[0, \varepsilon^2)$ and ensures that F becomes a part of the boundary of the unoriented moduli space. If F has codimension 2, then ν is a 2-plane bundle. The analysis is local so we can assume ν is a complex line bundle L . Identifying antipodal points is equivalent to passing to L^2 and we obtain a smooth structure on the quotient where the double cover ramifies over F . \square

3.3. The ramification set for the projection $\widetilde{\mathfrak{Z}}^+(p, q) \rightarrow \widetilde{\mathfrak{Z}}(p, q)$. Let P^+ belong to the fixed point set $\widetilde{\mathfrak{Z}}^+(p, q)$. We may choose real local coordinates $(\eta^1, \eta^2) \in (-\epsilon, \epsilon) \times (-\epsilon, \epsilon)$ which are centered at P so that $\widetilde{\mathfrak{Z}}^+(p, q)$ is defined by setting the real fiber coordinate $\eta^2 = 0$. Local coordinates for the corresponding point P in the boundary of $\widetilde{\mathfrak{Z}}(p, q)$ are then given by taking $(\eta^1, \eta^2) \in (-\epsilon, \epsilon) \times [0, \epsilon^2)$. The ramified double cover is the fold singularity given by $(\eta^1, \eta^2) \rightarrow (\eta^1, (\eta^2)^2)$.

3.4. The number of boundary components in $\mathfrak{Z}(p, q)$. Let $(p, q) = (1, 1)$ or $(p, q) = (0, 2)$ so the exceptional orbit plays no role. The structure group G_0 has two arc components corresponding to $\{a_{11} > 0, a_{22} > 0\}$ and $\{a_{11} < 0, a_{22} < 0\}$. Thus the number of boundary components in $\mathfrak{Z}(p, q)$ is half the number of components in $\mathcal{FZ}(p, q)$. We apply Equation (3.a) setting $c = b$ so $\rho = \text{diag}(a(d-b), b(b-d))$. The Ricci tensor is positive definite if $a(d-b) > 0$ and $b(b-d) > 0$. This gives rise to two components $\{a > 0, b < 0, d > b\}$ and $\{a < 0, b > 0, d < b\}$. Thus the boundary of $\mathfrak{Z}(0, 2)$ is connected; this is in agreement with Figure 1.1. On the other hand, the Ricci tensor is indefinite if $\{a(d-b) > 0, b(b-d) < 0\}$ or $\{a(d-b) < 0, b(b-d) > 0\}$. This gives rise to four components $\{a > 0, d > b, b > 0\}$ or $\{a < 0, d < b, b < 0\}$ or $\{a > 0, d > b, b < 0\}$ or $\{a < 0, d < b, b > 0\}$. Thus $\mathfrak{Z}_{1,1}$ has 2 boundary components. This again is in agreement with Figure 1.1. Finally, if we ignore the effect of the singular orbit, which we will treat in the next section, the same analysis shows $\mathfrak{Z}(2, 0)$ has one boundary component.

3.5. The ramification set for the projection $\widetilde{\mathfrak{W}}^+(p, q) \rightarrow \widetilde{\mathfrak{W}}(p, q)$. Let $B_\epsilon(0) := \{\eta \in \mathbb{C} : |\eta| < \epsilon\}$ be the open ball of radius ϵ in \mathbb{C} . Let P^+ belong to the fixed point set $\widetilde{\mathfrak{W}}^+(p, q)$. We may choose local complex coordinates $(\eta^1, \eta^2) \in B_\epsilon \times B_\epsilon$ which are centered at P^+ so that locally $\widetilde{\mathfrak{W}}^+(p, q)$ is defined by setting the complex fiber coordinate $\eta^2 = 0$. Local coordinates for the corresponding point P in $\widetilde{\mathfrak{W}}(p, q)$ are then given by taking $(\eta^1, \eta^2) \in B_\epsilon \times B_\epsilon$ and the ramified double cover is then the quadratic singularity given by $(\eta^1, \eta^2) \rightarrow (\eta^1, (\eta^2)^2)$.

4. THE ORBIFOLD STRUCTURE NEAR THE SINGULAR ORBIT $[\Gamma_0]$

4.1. Complex coordinates. We adopt the notation of Equation (2.b) and complexity:

$$\begin{aligned} f_1 &:= e_1 + \sqrt{-1}e_2, & f_2 &:= e_1 - \sqrt{-1}e_2, \\ f^1 &:= \tfrac{1}{2}(e^1 - \sqrt{-1}e^2), & f^2 &:= \tfrac{1}{2}(e^1 + \sqrt{-1}e^2). \end{aligned}$$

We identify $\mathbb{R}^8 = \mathbb{C}^4$ and define coordinates $\vec{\alpha} := (\alpha_{11}^1, \alpha_{11}^2, \alpha_{12}^1, \alpha_{12}^2) \in \mathbb{C}^4$ on $\mathcal{W}(2, 0)$ by defining $\tilde{\Gamma}(\vec{\alpha})$ to be:

$$\begin{aligned} \tilde{\Gamma}_{11}^1 &= \alpha_{11}^1, & \tilde{\Gamma}_{11}^2 &= \alpha_{11}^2, & \tilde{\Gamma}_{12}^1 &= \alpha_{12}^1, & \tilde{\Gamma}_{12}^2 &= \alpha_{12}^2, \\ \tilde{\Gamma}_{22}^2 &= \bar{\alpha}_{11}^1, & \tilde{\Gamma}_{22}^1 &= \bar{\alpha}_{11}^2, & \tilde{\Gamma}_{21}^2 &= \bar{\alpha}_{12}^1, & \tilde{\Gamma}_{21}^1 &= \bar{\alpha}_{12}^2. \end{aligned}$$

The singular orbit is then $\tilde{\Gamma}(0, 1, 0, 0)$; where we suppress normalizing constant of $1/\sqrt{2}$ as it plays no role. Similar coordinates on $\mathcal{Z}(2, 0)$ taking values in \mathbb{C}^3 are obtained by imposing the single condition

$$\alpha_{12}^1 = \tilde{\Gamma}_{12}^1 = \tilde{\Gamma}_{21}^1 = \bar{\alpha}_{12}^2.$$

We then have automatically $\tilde{\Gamma}_{12}^2 = \alpha_{12}^2 = \bar{\alpha}_{12}^1 = \tilde{\Gamma}_{21}^2$.

4.2. A complex representation of the general linear group. If $T \in \mathrm{GL}^+(2, \mathbb{R})$, then $T = T_{\beta_1, \beta_2}$ for $|\beta_1|^2 - |\beta_2|^2 > 0$ where

$$\begin{aligned} T_{\beta_1, \beta_2} f_1 &= \beta_1 f_1 + \beta_2 f_2, & T_{\beta_1, \beta_2} f_2 &= \bar{\beta}_2 f_1 + \bar{\beta}_1 f_2, \\ T_{\beta_1, \beta_2} f^1 &= \frac{1}{|\beta_1|^2 - |\beta_2|^2} (\bar{\beta}_1 f^1 - \bar{\beta}_2 f^2), & T_{\beta_1, \beta_2} f^2 &= \frac{1}{|\beta_1|^2 - |\beta_2|^2} (-\beta_2 f^1 + \beta_1 f^2). \end{aligned}$$

We wish to compute the tangent space to the orbit $C_0 := \mathrm{GL}^+(2, \mathbb{R}) \cdot \Gamma_0$. We consider the two curves $T_{1+t\beta, 0}$ and $T_{1, t\beta}$.

$$\begin{aligned} \{\partial_t T_{1+t\beta_1, 0} \Gamma_0|_{t=0}\}_{11}^1 &= 0, & \{\partial_t T_{1+t\beta_1, 0} \Gamma_0|_{t=0}\}_{11}^2 &= 3\beta_1, \\ \{\partial_t T_{1+t\beta_1, 0} \Gamma_0|_{t=0}\}_{12}^1 &= 0, & \{\partial_t T_{1+t\beta_1, 0} \Gamma_0|_{t=0}\}_{12}^2 &= 0, \\ \{\partial_t T_{1, t\beta_2} \Gamma_0|_{t=0}\}_{11}^1 &= -\bar{\beta}_2, & \{\partial_t T_{1, t\beta_2} \Gamma_0|_{t=0}\}_{11}^2 &= 0, \\ \{\partial_t T_{1, t\beta_2} \Gamma_0|_{t=0}\}_{12}^1 &= 0, & \{\partial_t T_{1, t\beta_2} \Gamma_0|_{t=0}\}_{12}^2 &= \bar{\beta}_2. \end{aligned}$$

Thus a transversal slice $s_{\mathcal{W}}(\alpha_1, \alpha_2)$ to C_0 in $\mathcal{W}(2, 0)$ can be taken to be:

$$\begin{aligned} s_{\mathcal{W}, 11}^1 &:= 0, & s_{\mathcal{W}, 11}^2 &:= 1, & s_{\mathcal{W}, 12}^1 &:= \bar{\alpha}_2, & s_{\mathcal{W}, 12}^2 &:= \alpha_1, \\ s_{\mathcal{W}, 22}^2 &:= 0, & s_{\mathcal{W}, 22}^1 &:= 1, & s_{\mathcal{W}, 21}^2 &:= \alpha_2, & s_{\mathcal{W}, 21}^1 &:= \bar{\alpha}_1. \end{aligned}$$

In defining the transversal slice $s_{\mathcal{Z}}(\alpha)$ over $\mathcal{Z}(2, 0)$, we set $\alpha := \alpha_1 = \alpha_2$ to ensure that $\tilde{\Gamma}_{12}^1 = \tilde{\Gamma}_{21}^1$ and $\tilde{\Gamma}_{12}^2 = \tilde{\Gamma}_{21}^2$:

$$\begin{aligned} s_{\mathcal{Z}, 11}^1 &:= 0, & s_{\mathcal{Z}, 11}^2 &:= 1, & s_{\mathcal{Z}, 12}^1 &:= \bar{\alpha}, & s_{\mathcal{Z}, 12}^2 &:= \alpha, \\ s_{\mathcal{Z}, 22}^2 &:= 0, & s_{\mathcal{Z}, 22}^1 &:= 1, & s_{\mathcal{Z}, 21}^2 &:= \alpha, & s_{\mathcal{Z}, 21}^1 &:= \bar{\alpha}. \end{aligned}$$

4.3. The proof of Theorem 1.5. Let $\lambda := e^{2\pi\sqrt{-1}/3}$. Define an action of \mathbb{Z}_3 by setting $T_\lambda f_1 := \lambda f_1$ and $T_\lambda f_2 := \bar{\lambda} f_2$. Then

$$\begin{aligned} T_\lambda \tilde{\Gamma}_{11}^1 &= \lambda \tilde{\Gamma}_{11}^1, & T_\lambda \tilde{\Gamma}_{11}^2 &= \tilde{\Gamma}_{11}^2, & T_\lambda \tilde{\Gamma}_{12}^1 &= \bar{\lambda} \tilde{\Gamma}_{12}^1, & T_\lambda \tilde{\Gamma}_{12}^2 &= \lambda \tilde{\Gamma}_{12}^2, \\ T_\lambda \tilde{\Gamma}_{22}^2 &= \bar{\lambda} \tilde{\Gamma}_{22}^2, & T_\lambda \tilde{\Gamma}_{22}^1 &= \tilde{\Gamma}_{22}^1, & T_\lambda \tilde{\Gamma}_{21}^2 &= \lambda \tilde{\Gamma}_{21}^2, & T_\lambda \tilde{\Gamma}_{21}^1 &= \bar{\lambda} \tilde{\Gamma}_{21}^1. \end{aligned}$$

The slices are equivariant with respect to this action, i.e.

$$T_\lambda s_{\mathcal{W}}(\alpha_1, \alpha_2) = s_{\mathcal{W}}(\lambda \alpha_1, \lambda \alpha_2) \text{ and } T_\lambda s_{\mathcal{Z}}(\alpha) = s_{\mathcal{Z}}(\lambda \alpha).$$

The slices projects down to define local coordinates on the oriented orbifolds where we must identify by the action of \mathbb{Z}^3 on \mathbb{C} when dealing with $\mathfrak{Z}^+(2, 0)$ and by the diagonal action of \mathbb{Z}^3 on \mathbb{C}^2 when dealing with $\mathfrak{W}^+(2, 0)$. This establishes Assertion 1 of Theorem 1.5.

Complex conjugation interchanges the roles of f_1 and f_2 and reverses the orientation. The slices are equivariant with respect to the action of complex conjugation

$$\bar{s}_{\mathcal{W}}(\alpha_1, \alpha_2) = s_{\mathcal{W}}(\bar{\alpha}_1, \bar{\alpha}_2) \text{ and } \bar{s}_{\mathcal{Z}}(\alpha) = s_{\mathcal{Z}}(\bar{\alpha}).$$

We adopt the notation of Definition 1.4 and let \mathbb{Z}_3 and complex conjugation generate the group s_3 which acts on \mathbb{C} and on \mathbb{C}^2 . The analysis of the orbifold structure performed in the orientable setting now extends to the non-orientable setting; the role that \mathbb{Z}_3 plays as the orbifold group in the orientable setting is now played by s_3 in the non-orientable setting. \square

4.4. The boundary of $\mathfrak{Z}(2, 0)$. We remark that the action of s_3 on \mathbb{C} gives rise to a corner with an angle of $\frac{2\pi}{3}$ at $[\Gamma_0] \in \mathfrak{Z}(2, 0)$. Furthermore, the orbifold singularity at $[\Gamma_0] \in \mathfrak{Z}^+(2, 0)$ can be eliminated by using coordinates $z \rightarrow z^3$; $\mathfrak{Z}^+(2, 0)$ has a smooth structure. Since the boundary of $\mathfrak{Z}(1, 1)$ and $\mathfrak{Z}(0, 2)$ is in fact smooth, Figure 1.1 is misleading in this regard; the apparent cusp is a function of the parametrization using (ψ_3, Ψ_3) and does not reflect the underlying topological structure.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge useful conversations on this subject with Professors M. Brozos-Vázquez, E. García-Río, O. Kowalski, and J. H. Park. Research partially supported by project GRC2013-045 (Spain).

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